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COMMUTATOR EQUATIONS IN FREE GROUPS

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ABSTRACT
Let $f_1, \cdots, f_n$ be free generators of a free group F. We consider the equation $[z_1, \cdots, z_n]_\omega = [f_1, \cdots, f_n]_{\omega'}$ where ω and ω' indicate the disposition of brackets in the higher commutators $[z_1, \cdots, z_n]_\omega$ and $[f_1, \cdots, f_n]_{\omega'}$. We give a necessary and sufficient condition on ω and ω' for the existence of solutions of this equation. It is also shown that for any solution $z_1 = r_1, \cdots, z_n = r_n$ we have $\langle r_1, \cdots, r_n \rangle = \langle f_1, \cdots, f_n \rangle$.

Introduction

From a well-known result of Nielsen [3] it follows that if for some elements r, s of a free group F on two generators a, b

$$[r, s] = [a, b]$$

then r, s freely generate F.

In the present paper similar properties of higher commutators are investigated. We recall the definition of a higher commutator.

Let θ be a free (non-associative) monoid on one generator i with respect to a bracket operation. θ is graded in a natural way, $\theta = \bigcup_{m=1}^{\infty} \theta_m$, where $\theta_1 = \{i\}$, $\theta_m \cap \theta_n = \varnothing$ for $m \neq n$ and for $\omega_1 \in \theta_m$, $\omega_2 \in \theta_n$, $[\omega_1, \omega_2] \in \theta_{m+n}$.

The higher commutator $[g_1, \cdots, g_n]_\omega$ of the type $\omega \in \theta_n$ of some elements $g_1, \cdots, g_n \in G$ is defined by induction on n as follows:

(i) if $n = 1$ then $\omega = i$ and $[g_1]_i = g_1$;

(ii) if $n > 1$ then $\omega = [\omega_1, \omega_2]$, $\omega_1 \in \theta_m$, $\omega_2 \in \theta_{n-m}$ and

$$[g_1, \cdots, g_n]_\omega = [[g_1, \cdots, g_m]_{\omega_1}, [g_{m+1}, \cdots, g_n]_{\omega_2}].$$

Consider also a free commutative monoid θ' on one generator i' and the homomorphism $\tau : \theta \to \theta'$ determined by $\tau : i \to i'$.

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Let $F_\infty = \langle f_1, f_2, \cdots \rangle$ be a free group of countable rank on free generators $f_1, f_2, \cdots$, and $F_n = \langle f_1, \cdots, f_n \rangle \leqq F_\infty$.

THEOREM 1. *The equation*

$$(*) \qquad [z_1, \cdots, z_n]_\omega = [f_1, \cdots, f_n]_{\omega'} \qquad (\omega, \omega' \in \theta_n)$$

in the free group F_∞ has a solution if and only if $\omega\tau = \omega'\tau$. If $r_1, \cdots, r_n \in F_\infty$ is a solution of equation (∗), *then $\langle r_1, \cdots, r_n \rangle = F_n$.*

The following lemma is the key statement in the proof of the theorem.

MAIN LEMMA. *Let a free group F be decomposed into a free product $F = H_1 * H_2$, and let $h_1 \in H_1$, $h_2 \in H_2$ be non-unit elements such that the cyclic groups $\langle h_1 \rangle$ and $\langle h_2 \rangle$ are isolated. If for some $g_1, g_2 \in F$, $[g_1, g_2] \neq 1$ and $[g_1, g_2] \in \langle h_1, h_2 \rangle$ then one of the following three cases holds*:

(i) *there exists an element $s \in \langle h_1, h_2 \rangle$ such that $s^{-1} g_1 s \in H_1$, $s^{-1} g_2 s \in H_1$ and $s^{-1}[g_1, g_2] s = h_1^{\pm 1}$;*

(ii) *there exists an element $t \in \langle h_1, h_2 \rangle$ such that*

$$t^{-1} g_1 t \in H_2, \quad t^{-1} g_2 t \in H_2 \quad \textit{and} \quad t^{-1}[g_1, g_2] t = h_2^{\pm 1};$$

(iii) $g_1, g_2 \in \langle h_1, h_2 \rangle$.

REMARK. The assumption that $\langle h_1 \rangle$ and $\langle h_2 \rangle$ are isolated cannot be omitted as the following example shows:

$$F = \langle f_1 \rangle * \langle f_2 \rangle, \quad h_1 = f_1, \quad h_2 = f_2^2, \quad [f_2 f_1, f_2^2] \in \langle f_1, f_2^2 \rangle \quad \text{but } f_2 f_1 \notin \langle f_1, f_2^2 \rangle.$$

Note that if r_1, r_2 are elements of a free group F then $\langle [r_1, r_2] \rangle$ is isolated [1].

In §1 we prove Theorem 1 using the Main Lemma and obtain a description of endomorphisms of F_∞ that fix $[f_1, \cdots, f_n]_\omega$ for $n > 2$. The §§2, 3, 4 are devoted to the proof of the Main Lemma.

§1. Proof of Theorem 1

The proof is based on separation of variables in the equation (∗) for $n > 2$. At first we need the following statement.

PROPOSITION. *Under the assumptions of the Main Lemma assume that $[g_1, g_2] = [h_1, h_2]$. If $g_1 = [g_3, g_4]$ then for some integers k, l one of the following cases holds*:

(1) $g_1 = [h_1, h_2]^{-k} h_1 [h_1, h_2]^k$,
$g_2 = [h_1, h_2]^{-k} h_1^l h_2 [h_1, h_2]^k$;

(2) $g_1 = [h_1, h_2]^{-k} h_1^{-1} h_2 h_1 [h_1, h_2]^k$,
$g_2 = [h_1, h_2]^{-k} (h_1^{-1} h_2 h_1)^l h_1^{-1} [h_1, h_2]^k$;
(3) $g_1 = [h_1, h_2]^{-k} h_2^{-1} h_1^{-1} h_2 [h_1, h_2]^k$,
$g_2 = [h_1, h_2]^{-k} (h_2^{-1} h_1^{-1} h_2)^l h_2^{-1} h_1^{-1} h_2^{-1} h_1 h_2 [h_1, h_2]^k$;
(4) $g_1 = [h_1, h_2]^{-k} h_2^{-1} [h_1, h_2]^k$,
$g_2 = [h_1, h_2]^{-k} h_2^{-l-1} h_1 h_2 [h_1, h_2]^k$.

If $g_2 = [g_5, g_6]$, *then for some integers* k, l *one of the following cases holds*:
(1′) $g_1 = [h_1, h_2]^{-k} h_2^l h_1 [h_1, h_2]^k$,
$g_2 = [h_1, h_2]^{-k} h_2 [h_1, h_2]^k$;
(2′) $g_1 = [h_1, h_2]^{-k} h_1^{-l-1} h_2 h_1 [h_1, h_2]^k$,
$g_2 = [h_1, h_2]^{-k} h_1^{-1} [h_1, h_2]^k$;
(3′) $g_1 = [h_1, h_2]^{-k} (h_2^{-1} h_1^{-1} h_2^{-1} h_1 h_2)^l h_2^{-1} h_1^{-1} h_2 [h_1, h_2]^k$,
$g_2 = [h_1, h_2]^{-k} h_2^{-1} h_1^{-1} h_2^{-1} h_1 h_2 [h_1, h_2]^k$;
(4′) $g_1 = [h_1, h_2]^{-k} (h_2^{-1} h_1 h_2)^l h_2^{-1} [h_1, h_2]^k$,
$g_2 = [h_1, h_2]^{-k} h_2^{-1} h_1 h_2 [h_1, h_2]^k$.

PROOF. It is enough to verify the first part of the proposition. $[h_1, h_2]$ is not conjugate to $h_1^{\pm}$ nor to $h_2^{\pm}$ in $\langle h_1, h_2 \rangle$. Therefore according to the Main Lemma it follows from $[g_1, g_2] = [h_1, h_2]$ that $g_1, g_2 \in \langle h_1, h_2 \rangle$, and, by the theorem of Nielsen, g_1 and g_2 freely generate $\langle h_1, h_2 \rangle$. In particular, $g_1, g_2 \notin (\langle h_1, h_2 \rangle)'$. Applying once more the Main Lemma we obtain from $g_1 = [g_3, g_4] \in \langle h_1, h_2 \rangle$ that g_1 is conjugated in $\langle h_1, h_2 \rangle$ to one of the elements $h_1, h_2, h_1^{-1}, h_2^{-1}$.

We have $[g_1, g_2] = [h_1, h_2] = [h_2^{-1} h_1^{-1} h_2, h_2^{-1} h_1^{-1} h_2^{-1} h_1 h_2] = [h_1^{-1} h_2 h_1, h_1^{-1}] = [h_2^{-1}, h_2^{-1} h_1 h_2]$, and each pair of elements (h_1, h_2), $(h_2^{-1} h_1^{-1} h_2, h_2^{-1} h_1^{-1} h_2^{-1} h_1 h_2)$, $(h_1^{-1} h_2 h_1, h_1^{-1})$, $(h_2^{-1}, h_2^{-1} h_1 h_2)$ generates $\langle h_1, h_2 \rangle$. Therefore we may complete the proof of the proposition by proving the following assertion:

If S is a free group on free generators s_1, s_2, $t, t_1, t_2 \in S$, $t_1 = t^{-1} s_1 t$ and $[t_1, t_2] = [s_1, s_2]$, then for some integers k and l

$$t_1 = [s_1, s_2]^{-k} s_1 [s_1, s_2]^k, \qquad t_2 = [s_1, s_2]^{-k} s_1^l s_2 [s_1, s_2]^k.$$

Indeed, we have $t_1 \equiv s_1 \pmod{S'}$ and $t_2 \equiv s_1^l s_2^m \pmod{S'}$. Let φ be an endomorphism of S defined by $s_1 \varphi = t_1$, $s_2 \varphi = t_1^{-l} t_2$. In S/S', φ induces a linear mapping with a matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right)$. As $[t_1, t_1^{-l} t_2] = [t_1, t_2] = [s_1, s_2]$, according to theorem 3 [2], $(t_1, t_1^{-1} t_2)$ is a positive pair of generators of S. Hence $\det\left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right) = 1$, $m = 1$. Thus φ is an automorphism of S (identical modulo S'). Then by [1, p. 466] φ is an inner automorphism of S. It means that for some $s \in S$, $t_1 = s^{-1} s_1 s$ and $t_1^{-l} t_2 = s^{-1} s_2 s$. We have $[s_1, s_2] = [t_1, t_1^{-l} t_2] = s^{-1} [s_1, s_2] s$; therefore $s = [s_1, s_2]^k$ for some integer k, and hence

$$t_1 = [s_1, s_2]^{-k} s_1 [s_1, s_2]^k \quad \text{and} \quad t_2 = [s_1, s_2]^{-k} s_1^l s_2 [s_1, s_2]^k,$$

as required.

Now we return to the proof of Theorem 1. In the case $n = 1$, the assertion is trivial, and for $n = 2$ it follows from [3]. Let $n > 2$.

We have $\omega = [\omega_1, \omega_2]$ and $\omega' = [\omega_1', \omega_2']$, where $\omega_1 \in \theta_m$, $\omega_2 \in \theta_{n-m}$, $\omega_1' \in \theta_{m'}$, $\omega_2' \in \theta_{n-m'}$, and, correspondingly,

$$[z_1, \cdots, z_n]_\omega = [[z_1, \cdots, z_m]_{\omega_1}, [z_{m+1}, \cdots, z_n]_{\omega_2}],$$

$$[f_1, \cdots, f_n]_{\omega'} = [[f_1, \cdots, f_{m'}]_{\omega_1'}, [f_{m'+1}, \cdots, f_n]_{\omega_2'}].$$

If we set $h_1 = [f_1, \cdots, f_{m'}]_{\omega_1'}$, $h_2 = [f_{m'+1}, \cdots, f_n]_{\omega_2'}$,

$$H_1 = F_{m'} = \langle f_1, \cdots, f_{m'} \rangle, \qquad H_2 = \langle f_{m'+1}, \cdots, f_n, \cdots \rangle,$$

then $h_1 \in H_1$, $h_2 \in H_2$, and $F_\infty = H_1 * H_2$. The conditions of the Main Lemma are satisfied. Let $w_1 = [z_1, \cdots, z_m]_{\omega_1}$ and $w_2 = [z_{m+1}, \cdots, z_n]_{\omega_2}$. The equation (*) can be rewritten as

$$(**) \qquad [w_1, w_2] = [h_1, h_2].$$

I. If $m = 1$ then $n - m \geqq 2$, whence $\omega_2 = [\omega_5, \omega_6]$ and $w_2 = [w_5, w_6]$ for some w_5 and w_6. According to the Proposition the equation (**) is equivalent to a disjunction of the following four systems of equations, in which k and l are arbitrary integers:

$$(1) \quad \begin{cases} w_1 = z_1 = [h_1, h_2]^{-k} h_2^l h_1 [h_1, h_2]^k, \\ w_2 = [z_2, \cdots, z_n]_{\omega_2} = [h_1, h_2]^{-k} [f_{m+1}, \cdots, f_n]_{\omega_2'} [h_1, h_2]^k; \end{cases}$$

$$(2) \quad \begin{cases} w_1 = z_1 = [h_1, h_2]^{-k} h_1^{-l-1} h_2 h_1 [h_1, h_2]^k, \\ w_2 = [z_2, \cdots, z_n]_{\omega_2} = [h_1, h_2]^{-k} [f_1, \cdots, f_{m'}]_{\omega_1'}^{-1} [h_1, h_2]^k; \end{cases}$$

$$(3) \quad \begin{cases} w_1 = z_1 = [h_1, h_2]^{-k} h_2^{-1} h_1^{-1} h_2^{-l+1} [h_1, h_2]^k, \\ w_2 = [z_2, \cdots, z_n]_{\omega_2} = [h_1, h_2]^{-k} h_2^{-1} h_1^{-1} [f_{m'+1}, \cdots, f_n]_{\omega_2'}^{-1} h_1 h_2 [h_1, h_2]^k; \end{cases}$$

$$(4) \quad \begin{cases} w_1 = z_1 = [h_1, h_2]^{-k} h_2^{-1} h_1^l [h_1, h_2]^k, \\ w_1 = [z_2, \cdots, z_n]_{\omega_2} = [h_1, h_2]^{-k} h_2^{-1} [f_1, \cdots, f_{m'}]_{\omega_1'} h_2 [h_1, h_2]^k. \end{cases}$$

II. If $1 < m < n - 1$, then $\omega_1 = [\omega_3, \omega_4]$, $\omega_2 = [\omega_5, \omega_6]$, and $w_1 = [w_3, w_4]$, $w_2 = [w_5, w_6]$ for some w_3, w_4, w_5, and w_6. Using the Proposition, we obtain that in this case the equation (**) is equivalent to a disjunction of the following four systems of equations, in which k is an arbitrary integer:

$$(5)\qquad \begin{cases} w_1 = [z_1, \cdots, z_m]_{\omega_1} = [h_1, h_2]^{-k}[f_1, \cdots, f_{m'}]_{\omega_1'}[h_1, h_2]^k, \\ w_2 = [z_{m+1}, \cdots, z_n]_{\omega_2} = [h_1, h_2]^{-k}[f_{m'+1}, \cdots, f_n]_{\omega_2'}[h_1, h_2]^k; \end{cases}$$

$$(6)\qquad \begin{cases} w_1 = [z_1, \cdots, z_m]_{\omega_1} = [h_1, h_2]^{-k}h_1^{-1}[f_{m'+1}, \cdots, f_n]_{\omega_2'}h_1[h_1, h_2]^k, \\ w_2 = [z_{m+1}, \cdots, z_n]_{\omega_2} = [h_1, h_2]^{-k}[f_1, \cdots, f_{m'}]_{\omega_1'}^{-1}[h_1, h_2]^k; \end{cases}$$

$$(7)\qquad \begin{cases} w_1 = [z_1, \cdots, z_m]_{\omega_1} = [h_1, h_2]^{-k}h_2^{-1}[f_1, \cdots, f_{m'}]_{\omega_1'}^{-1}h_2[h_1, h_2]^k, \\ w_2 = [z_{m+1}, \cdots, z_n]_{\omega_2} = [h_1, h_2]^{-k}h_2^{-1}h_1^{-1}[f_{m'+1}, \cdots, f_n]_{\omega_2'}^{-1}h_1h_2[h_1, h_2]^k; \end{cases}$$

$$(8)\qquad \begin{cases} w_1 = [z_1, \cdots, z_m]_{\omega_1} = [h_1, h_2]^{-k}[f_{m'+1}, \cdots, f_n]_{\omega_2'}^{-1}[h_1, h_2]^k, \\ w_2 = [z_{m+1}, \cdots, z_n]_{\omega_2} = [h_1, h_2]^{-k}h_2^{-1}[f_1, \cdots, f_{m'}]_{\omega_1'}h_2[h_1, h_2]^k. \end{cases}$$

III. If $m = n - 1$, then $\omega_1 = [\omega_3, \omega_4]$ and again, according to the Proposition, the equation (**) is equivalent to a disjunction of the following four systems of equations in which k and l are arbitrary integers:

$$(9)\qquad \begin{cases} w_1 = [z_1, \cdots, z_{n-1}]_{\omega_1} = [h_1, h_2]^{-k}[f_1, \cdots, f_{m'}]_{\omega_1'}[h_1, h_2]^k, \\ w_2 = z_n = [h_1, h_2]^{-k}h_1^{l}h_2[h_1, h_2]^k; \end{cases}$$

$$(10)\qquad \begin{cases} w_1 = [z_1, \cdots, z_{n-1}]_{\omega_1} = [h_1, h_2]^{-k}h_1^{-1}[f_{m'+1}, \cdots, f_n]_{\omega_2'}h_1[h_1, h_2]^k, \\ w_2 = z_n = [h_1, h_2]^{-k}h_1^{-1}h_2^{-l}[h_1, h_2]^k; \end{cases}$$

$$(11)\qquad \begin{cases} w_1 = [z_1, \cdots, z_{n-1}]_{\omega_1} = [h_1, h_2]^{-k}h_2^{-1}[f_1, \cdots, f_{m'}]_{\omega_1'}^{-1}h_2[h_1, h_2]^k, \\ w_2 = z_n = [h_1, h_2]^{-k}h_2^{-1}h_1^{-l-1}h_2^{-1}h_1h_2[h_1, h_2]^k; \end{cases}$$

$$(12)\qquad \begin{cases} w_1 = [z_1, \cdots, z_{n-1}]_{\omega_1} = [h_1, h_2]^{-k}[f_{m'+1}, \cdots, f_n]_{\omega_2'}^{-1}[h_1, h_2]^k, \\ w_2 = z_n = [h_1, h_2]^{-k}h_2^{-l-1}h_1h_2[h_1, h_2]^k. \end{cases}$$

Comparing the weights of commutators in the left and right parts of systems of equations (1)–(12), we see that the systems (1), (3), (5), (7), (9), (11) do not have solutions for $m \neq m'$, but the systems (2), (4), (6), (8), (10), (12) do not have solutions for $m \neq n - m'$. Therefore it is enough for the systems (2), (4), (6), (8), (10), (12) to consider the case $m = n - m'$.

Then for every system (i), $1 \leqq i \leqq 12$, and for every value of k and l, it is possible to construct an automorphism $\varphi = \varphi_i(k, l)$ of the group F_∞ such that the subgroup F_n is φ-invariant and the system (i) can be rewritten as

$$(***)\qquad \begin{aligned} [z_1, \cdots, z_m]_{\omega_1} &= [f_1\varphi, \cdots, f_m\varphi]_{\omega_1^*}, \\ [z_{m+1}, \cdots, z_n]_{\omega_2} &= [f_{m+1}\varphi, \cdots, f_n\varphi]_{\omega_2^*}, \end{aligned}$$

where $\omega_1^*\tau = \omega_1'\tau$ and $\omega_2^*\tau = \omega_2'\tau$ for $i = 1,3,5,7,9,11$, and $\omega_1^*\tau = \omega_2'\tau$, $\omega_2^*\tau = \omega_1'\tau$ for $i = 2,4,6,8,10,12$.

For example, for $i = 7$, we have $m = m'$, $\omega_1' = [\omega_3', \omega_4']$, $\omega_2' = [\omega_5', \omega_6']$, where $\omega_3' \in \theta_{m_1}$, $\omega_4' \in \theta_{m-m_1}$, $\omega_5' \in \theta_{m_2}$, $\omega_6' \in \theta_{n-m-m_2}$, and we set $\omega_1^* = [\omega_4', \omega_3']$, $\omega_2^* = [\omega_6', \omega_5']$ and $\varphi = \mu\nu$, where

$$(13)\qquad \begin{cases} f_j\mu = f_{j+m_1} & \text{if } 1 \leqq j \leqq m - m_1, \\ f_j\mu = f_{j-m+m_1} & \text{if } m - m_1 + 1 \leqq j \leqq m, \\ f_j\mu = f_{j+m_2} & \text{if } m + 1 \leqq j \leqq n - m_2, \\ f_j\mu = f_{j-n+m+m_2} & \text{if } n - m_2 + 1 \leqq j \leqq n, \\ f_j\mu = f_j & \text{if } j > n; \end{cases}$$

$$(14)\qquad \begin{cases} f_j\nu = [h_1, h_2]^{-k} h_2^{-1} f_j h_2 [h_1, h_2]^k & \text{if } 1 \leqq j \leqq m, \\ f_j\nu = [h_1, h_2]^{-k} h_2^{-1} h_1^{-1} f_j h_1 h_2 [h_1, h_2]^k & \text{if } j > m. \end{cases}$$

As $F_\infty = H_1 * H_2 = H_1 * h_1^{-1}H_2h_1 = [h_1, h_2]^{-k}h_2^{-1}(H_1 * h_1^{-1}H_2h_1)h_2[h_1, h_2]^k$, then ν is in fact an automorphism. It is obvious that F_1 is μ-invariant and ν-invariant. Using (13) and (14) one can rewrite (7) in the form (***). Other cases are similar.

We now proceed by induction on n. Assume that for all $n < n_0$ the theorem is already proved. We prove it for $n = n_0$, where $n_0 > 2$.

If the equation (*) has a solution $z_1 = r_1, \cdots, z_n = r_n$, then for some automorphism φ this solution is also a solution of the system of equations (***). Since $m < n = n_0$ and $n - m < n = n_0$, by the induction hypothesis

$$\omega_1\tau = \omega_1^*\tau, \quad \omega_2\tau = \omega_2^*\tau, \quad \langle r_1, \cdots, r_m \rangle = \langle f_1\varphi, \cdots, f_m\varphi \rangle,$$

and

$$\langle r_{m+1}, \cdots, r_n \rangle = \langle f_{m+1}\varphi, \cdots, f_n\varphi \rangle,$$

whence

$$\omega\tau = [\omega_1\tau, \omega_2\tau] = [\omega_1^*\tau, \omega_2^*\tau] = [\omega_1'\tau, \omega_2'\tau] = \omega'\tau$$

and

$$\langle r_1, \cdots, r_n \rangle = \langle f_1\varphi, \cdots, f_n\varphi \rangle = F_n\varphi = F_n.$$

Now let $\omega\tau = \omega'\tau$. Then $\omega_1\tau = \omega_1'\tau$ and $\omega_2\tau = \omega_2'\tau$, or $\omega_1\tau = \omega_2'\tau$ and $\omega_2\tau = \omega_1'\tau$.

If $\omega_1\tau = \omega_1'\tau$ and $\omega_2\tau = \omega_2'\tau$ then the system (***) has a solution provided $\omega_1^*\tau = \omega_1'\tau$ and $\omega_2^*\tau = \omega_1'\tau$. In this case for $m = 1$ (respectively, for $1 < m < n-1$, or for $m = n-1$) the systems (1) and (3) (respectively, (5) and (7), or (9) and (11)) have solutions, and, therefore, the equation (*) has a solution.

If $\omega_1\tau = \omega_2'\tau$ and $\omega_2\tau = \omega_1'\tau$ then the system (***) has a solution provided $\omega_1^*\tau = \omega_2'\tau$ and $\omega_2^*\tau = \omega_1'\tau$. In this case for $m = 1$ (respectively, for $1 < m < n-1$, or for $m = n-1$) the systems (2) and (4) (respectively, (6) and (8), or (10) and (12)) have solutions, and, therefore, the equation (*) has a solution.

This completes the proof of the theorem.

Notice that if $z_1 = r_1, \cdots, z_n = r_n$ is a solution of the equation (*) and ψ is an endomorphism of F_∞ such that $f_i\psi = r_i$, $1 \leqq i \leqq n$, then $[f_1, \cdots, f_n]_\omega\psi = [f_1, \cdots, f_n]_{\omega'}$. Therefore setting $\omega = \omega'$ we obtain from the proof of Theorem 1 the following assertion.

THEOREM 2. *Let* $n > 2$, $\omega \in \theta_n$, $\omega = [\omega_1, \omega_2]$, $\omega_1 \in \theta_m$, $\omega_2 \in \theta_{n-m}$, $h_1 = [f_1, \cdots, f_m]_{\omega_1}$, $h_2 = [f_{m+1}, \cdots, f_n]_{\omega_2}$, $h = [h_1, h_2] = [f_1, \cdots, f_n]_\omega$, *and let* ψ *be an automorphism of* F_∞ *such that* $h\psi = \psi$. *Then*:

(1) *For* $1 < m < n-1$ *and* $\omega_1\tau = \omega_2\tau$ *there are four possibilities*:

(i) $h_1\psi = h^{-k}h_1h^k$, $h_2\psi = h^{-k}h_2h^k$;

(ii) $h_1\psi = h^{-k}h_1h^k$, $h_2\psi = h^{-k}h_2^{-1}h_1^{-1}h_2^{-1}h_1h_2h_2h^k$;

(iii) $h_1\psi = h^{-k}h_1^{-1}h_1^{-1}h_2h^k$, $h_2\psi = h^{-k}h_2^{-1}h_1^{-1}h_2^{-1}h_1h_2h^k$;

(iv) $h_1\psi = h^{-k}h_2^{-1}h^k$, $h_2\psi = h^{-k}h_2^{-1}h_1h_2h^k$.

(2) *For* $1 < m < n-1$ *and* $\omega_1\tau \neq \omega_2\tau$ *there are the possibilities* (i) *and* (ii) *from the preceding point.*

(3) *For* $m = 1$ *there are two possibilities*:

(i) $h_1\psi = h^{-k}h_2^lh_1h^k$, $h_2\psi = h^{-k}h_2h^k$;

(ii) $h_1\psi = h^{-k}h_2^{-1}h_1^{-1}h_2^{-l+1}h^k$, $h_2\psi = h^{-k}h_2^{-1}h_1^{-1}h_2^{-1}h_1h_2h^k$.

(4) *For* $m = n-1$ *there are two possibilities*:

(i) $h_1\psi = h^{-k}h_1h^k$, $h_2\psi = h^{-k}h_1^lh_2h^k$;

(ii) $h_1\psi = h^{-k}h_2^{-1}h_1^{-1}h_2h^k$, $h_2\psi = h^{-k}h_2^{-1}h_1^{-l-1}h_1^{-1}h_1h_2h^k$,

where k *and* l *are arbitrary integers.*

For $\omega \in \theta_n$ and for elements $g_1, \cdots, g_n \in G$ let the set of elements $\Lambda_\omega(g_1, \cdots, g_n)$ be defined as follows: $\Lambda_i(g_1) = \emptyset$, and if $n > 1$, $\omega = [\omega_1, \omega_2]$, $\omega_1 \in \theta_m$, $\omega_2 \in \theta_{n-m}$

$$\Lambda_\omega(g_1, \cdots, g_n) = \Lambda_{\omega_1}(g_1, \cdots, g_m) \cup \Lambda_{\omega_2}(g_{m+1}, \cdots, g_n)$$

$$\cup \{[g_1, \cdots, g_n]_\omega, [g_1, \cdots, g_n]_\omega^{-1}\}.$$

COROLLARY 1. *Let ψ be an endomorphism of F_∞ such that $[f_1,\cdots,f_n]_\omega\psi = [f_1,\cdots,f_n]_\omega^{\pm 1}$. Then for every $g \in \Lambda_\omega(f_1,\cdots,f_n)$, $g\psi$ is conjugate to some $g' \in \Lambda_\omega(f_1,\cdots,f_n)$.*

PROOF. We proceed by induction on n. For $n = 1$ the assertion is trivial and suppose it is proved for all integers $< n$. If $[f_1,\cdots,f_n]_\omega\psi = [f_1,\cdots,f_n]_\omega$, then Theorem 2 gives the statement. Let $[f_1,\cdots,f_n]_\omega\psi = [f_1,\cdots,f_n]_\omega^{-1}$. By Theorem 1 there exists an endomorphism ξ of F_∞ such that

$$[f_1,\cdots,f_m]_{\omega_1}\xi = [f_{m+1},\cdots,f_n]_{\omega_2}^{-1}[f_1,\cdots,f_m]_{\omega_1}[f_{m+1},\cdots,f_n]_{\omega_2},$$
$$[f_{m+1},\cdots,f_n]_{\omega_2}\xi = [f_{m+1},\cdots,f_n]_{\omega_2}^{-1}.$$

Then $[f_1,\cdots,f_n]_\omega\xi = [f_1,\cdots,f_n]_\omega^{-1}$ and $[f_1,\cdots,f_n]_\omega\xi\psi = [f_1,\cdots,f_n]_\omega$. By induction hypothesis ξ has the required property, therefore ψ also has it.

Let A_n be the group of automorphisms of the group $F_n = \langle f_1,\cdots,f_n\rangle$ and let
$G_\omega = C_{A_n}([f_1,\cdots,f_n]_\omega)$,

$$H_1 = C_{A_n}([f_1,\cdots,f_m]_{\omega_1}, f_{m+1},\cdots,f_n), \quad H_2 = C_{A_n}(f_1,\cdots,f_m,[f_{m+1},\cdots,f_n]_{\omega_2}).$$

COROLLARY 2. *Under the assumption of Theorem 2 for $1 < m < n-1$ and $\omega_1\tau = \omega_2\tau$ there is an exact sequence*

$$1 \to H_1 \times H_2 \times Z \to G_\omega \to Z_4 \to 1;$$

for $1 < m < n-1$ and $\omega_1\tau \neq \omega_2\tau$ there is an exact sequence

$$1 \to H_1 \times H_2 \times Z \to G_\omega \to Z_2 \to 1,$$

for $m = 1$ there is an exact sequence

$$1 \to Z \times H_2 \times Z \to G_\omega \to Z_2 \to 1,$$

and for $m = n-1$ there is an exact sequence

$$1 \to H_1 \times Z \times Z \to G_\omega \to Z_2 \to 1.$$

This follows from Theorem 2 by an immediate calculation.

A description of $G_{[i,i]}$ is given in [2]. It follows from theorem 1 [2] that $G_{[i,i]}$ is generated by the automorphisms λ,μ of F_2 where $f_1\lambda = f_2^{-1}$, $f_2\lambda = f_2^{-1}f_1$, $f_1\mu = f_2f_1$, $f_2\mu = f_2$. Here $\lambda^2 = (\lambda\mu)^3$, and λ^4 is an inner automorphism:

$$f_i\lambda^4 = [f_1,f_2]^{-1}f_i[f_1,f_2], \qquad i = 1,2.$$

The following diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \langle X^4\rangle & \longrightarrow & \langle X, Y \mid X^2 = (XY)^3\rangle & \xrightarrow{u} & \mathrm{SL}_2(Z) & \longrightarrow & 1 \\
 & & \cong\big\downarrow & & \rho\big\downarrow & & \big\| & & \\
1 & \longrightarrow & \langle\lambda^4\rangle & \longrightarrow & G_{[i,i]} & \xrightarrow{v} & \mathrm{SL}_2(Z) & \longrightarrow & 1
\end{array}$$

is commutative, where $X\rho = \mu$, $Y\rho = \mu$, $Xu = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, $Yu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and for $\varphi \in G_{[i,i]}$, φv is the matrix (with respect to the basis f_1F_2', f_2F_2') of the automorphism of F_2/F_2', induced by φ. It is well-known that the upper row is exact, therefore the lower row is exact and ρ is an isomorphism.

We define functions $\alpha(\omega)$, $\beta(\omega)$, $\gamma(\omega)$, $\delta(\omega)$ as follows:

$$\alpha(i) = 0, \quad \beta(i) = 1, \quad \gamma(i) = 0, \quad \delta(i) = 0, \quad \alpha([i, i]) = 1,$$

$$\beta([i, i]) = \gamma([i, i]) = \delta([i, i]) = 0,$$

and for $\omega \in \theta_n$, $n > 2$, $\omega = [\omega_1, \omega_2]$, $\omega_1 \in \theta_m$, $\omega_2 \in \theta_{n-m}$

$$\alpha(\omega) = \alpha(\omega_1) + \alpha(\omega_2), \qquad \beta(\omega) = \beta(\omega_1) + \beta(\omega_2),$$

$$\gamma(\omega) = \begin{cases} \gamma(\omega_1) + \gamma(\omega_2) + 1, & \text{if } \omega_1\tau \neq \omega_2\tau, \\ \gamma(\omega_1) + \gamma(\omega_2) & \text{if } \omega_1\tau = \omega_2\tau, \end{cases}$$

$$\delta(\omega) = \begin{cases} \delta(\omega_1) + \delta(\omega_2), & \text{if } \omega_1\tau \neq \omega_2\tau, \\ \delta(\omega_1) + \delta(\omega_2) + 1 & \text{if } \omega_1\tau = \omega_2\tau. \end{cases}$$

Let H_ω be a subgroup of G_ω consisting of all $\varphi \in G_\omega$ such that $g\varphi$ is conjugate to g for all $g \in \Lambda_\omega(f_1, \cdots, f_n)$.

COROLLARY 3. *H_ω is normal in G_ω, $H_\omega \cong G_{[i,i]}^{\alpha(\beta)} \times Z^{\beta(\omega)+n-2}$, G_ω / H_ω is finite and of order* $2^{\gamma(\omega)+2\delta(\omega)}$.

This corollary easily follows from the preceding results using induction on the weight of ω.

§2. Irreducible form of a commutator

Let M be a set and x_α and y_α symbols with $\alpha \in M$. Denote by Y the semigroup of words in the alphabet $\{x_\alpha, y_\alpha \mid \alpha \in M\}$, and let F be a free group on generators $f_\alpha, \alpha \in M$. Let $\xi : Y \to F$ be a homomorphism of semigroups such that $x_\alpha\xi = f_\alpha$, $y_\alpha\xi = f_\alpha^{-1}$. Let further $\sigma : Y \to Y$ be an antiisomorphism, defined by $x_\alpha\sigma = y_\alpha$, $y_\alpha\sigma = x_\alpha$, $\alpha \in M$. For every $w \in Y$, $\omega\sigma\xi = (\omega\xi)^{-1}$.

A word from Y is called irreducible if the symbols x_α and y_α do not appear in it one near the other for all $\alpha \in M$. It is well known that the restriction of ξ on the set X of irreducible words is bijective; therefore there exists an inverse mapping $\Theta : F \to X$.

A word $\omega \in Y$ is said to be cyclically reducible if for some $\alpha \in M$ it begins with x_α and ends with y_α, or begins with y_α and ends with x_α. Otherwise it is said to be cyclically irreducible.

The length of a word $w \in Y$ will be denoted as $l(w)$. The word $w\sigma$ we shall write also $\bar{w}$. The graphical equality of words will be denoted by the symbol $\equiv$.

Let us consider in Y the subsets Δ_i, $1 \leqq i \leqq 8$, defined as follows:

for some word $w \in Y$ we set

$w \in \Delta_1$, if $w \equiv \bar{c}\bar{d}cd$;

$w \in \Delta_2$, if $w \equiv \bar{b}\bar{c}\bar{d}bcd$;

$w \in \Delta_3$, if $w \equiv \bar{b}\bar{c}b\bar{e}ce$;

$w \in \Delta_4$, if $w \equiv \bar{b}\bar{c}\bar{d}b\bar{e}cde$;

$w \in \Delta_5$, if $w \equiv \bar{a}\bar{c}\bar{d}cda$;

$w \in \Delta_6$, if $w \equiv \bar{a}\bar{b}\bar{c}\bar{d}bcda$;

$w \in \Delta_7$, if $w \equiv \bar{a}\bar{b}\bar{c}b\bar{e}cea$;

$w \in \Delta_8$, if $w \equiv \bar{a}\bar{b}\bar{c}\bar{d}b\bar{e}cdea$,

where a, b, c, d, e denote non-empty words from Y whenever they appear.

Let $\Delta = \bigcup_{i=1}^{8} \Delta_i$. Wicks [5] has proved the following statement.

LEMMA 1. *A non-unit element $r \in F$ is a commutator of some $s_1, s_2 \in F$ if and only if for some $q \in F$ $(q^{-1}rq)\Theta \equiv \bar{a}\bar{b}\bar{c}abc$ where some of the words a, b, c may be empty.*

Since all cyclic permutations of a word $\bar{a}\bar{b}\bar{c}abc$ belong to $\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$, we obtain:

LEMMA 2. *A non-unit element $r \in F$ is a commutator of some $s_1, s_2 \in F$ if and only if $r\Theta \in \Delta$.*

§3. Some further lemmas

Let $M = M_1 \cup M_2$, $M_1 \cap M_2 = \varnothing$, $H_1 = \langle f_\alpha, \alpha \in M_1 \rangle$, $H_2 = \langle f_\alpha, \alpha \in M_2 \rangle$, $X_1 = H_1\Theta$, $X_2 = H_2\Theta$.

Further, suppose that $M_1 \neq \varnothing$, $M_2 \neq \varnothing$, and that v_1 and v_2 are some non-empty cyclically reduced words, $v_1 \in X_1$, $v_2 \in X_2$. Let $V = (\langle v_1\xi, v_2\xi \rangle)\Theta$.

LEMMA 3. *If $\bar{a}ba \in V$ and the word b is cyclically irreducible, then $a, b \in V$.*

PROOF. We may assume that $a \not\equiv 1$ and $b \not\equiv 1$. For some words q_1, q_2, q_3, a_1, c_1, c_2 we have

$$\bar{a}ba \equiv c_1q_2q_1a_1 \equiv \bar{a}_1\bar{q}_1\bar{q}_3c_2,$$

where $a_1 \in V$, $q_1a_1 \equiv a$, $q_2q_1a_1 \in V$, $\bar{a}_1\bar{q}_1\bar{q}_3 \in V$, and each of the words q_1q_2, q_1q_3 coincides with one of the words $v_1, \bar{v}_1, v_2, \bar{v}_2$. If $q_1 \not\equiv 1$ then $q_2 \equiv q_3$, hence $q_2 \equiv 1$

because the word b is cyclically reduced. Thus, in such a way or another, $a \equiv q_1 a_1 \in V$. Then also $b \in V$. The lemma is proved.

LEMMA 4. *If $a, b \in V$, $a \equiv a_1 p$, $b \equiv \bar{p} b_1$ and $a_1, b_1 \in X$, then $a_1, p, b_1 \in V$.*

PROOF. If one of the words a_1, b_1 is empty, then the assertion is obvious. Assume that $a_1 \not\equiv 1$ and $b_1 \not\equiv 1$. Let us consider the word $\bar{p} b_1 c a_1 p$, where

$c \equiv 1$, if the word $a_1 b_1$ is cyclically irreducible,

$c \equiv v_2$, if $a_1 b_1$ is cyclically reducible, $a \equiv v_1 a_2$ or $a \equiv \bar{v}_1 a_2$,

$c \equiv v_1$, if $a_1 b_1$ is cyclically reducible, $a \equiv v_2 a_3$ or $a \equiv \bar{v}_2 a_3$.

In each of these three cases $\bar{p} b_1 c a_1 p \in X$, and the word $b_1 c a_1$ is cyclically irreducible. By Lemma 3, $p \in V$ and, therefore, $a_1 \in V$ and $b_1 \in V$. The lemma is proved.

A word w is called simple if it cannot be represented in the form $v \equiv u^k$ with $k > 1$ (see [4], definition 22). From now on we suppose that the words v_1 and v_2 are simple.

LEMMA 5. *If $a v_1 b \in V$, or $a v_2 b \in V$, then $a, b \in V$.*

PROOF. Let $a v_1 b \in V$. We have $a v_1 b \equiv a_1 a_2 v_1 b_2 b_1$, where a_1, $a_2 v_1 b_2$, $b_1 \in V$ and $a_2 v_1 b_2 \in X_1$. Then for some $k \geqq 1$, $a_2 v_1 b_2 \equiv v_1^k$, or $a_2 v_1 b_2 \equiv \bar{v}_1^k$. But the equality $a_2 v_1 b_2 \equiv \bar{v}_1^k$ is impossible, because v_1 cannot coincide with a cyclic permutation of $\bar{v}_1$ (see [4], definition 14 and lemma 45).

Since v_1 is simple, it cannot coincide with its non-trivial cyclic permutation ([4], lemma 20), therefore it follows from $a_2 v_1 b_2 \equiv v_1^k$ that $a_2 \equiv v_1^{i-1}$ and $b_2 \equiv v_1^{k-i}$ for some i, $1 \leqq i \leqq k$. Hence we obtain $a \in V$ and $b \in V$. The case $a v_2 b \in V$ is similar. The lemma is proved.

LEMMA 6. *If $ab \in V$ and $ba \in V$, then $a \in V$ and $b \in V$.*

PROOF. We may assume that $ab \not\equiv 1$. Then $ab \equiv u_1 c_1 \equiv c_2 u_2$, where each of the words u_1, u_2 coincides with one of the words $v_1, \bar{v}_1, v_2, \bar{v}_2$. If $l(u_1) \leqq l(a)$, then $a \equiv u_1 a_1$, therefore $ba \equiv b u_1 a_1 \in V$, whence by Lemma 5 $b \in V$ and, further, $a \in V$. Similarly, if $l(u_2) \leqq l(b)$, then $a \in V$ and $b \in V$. But if $l(u_1) > l(a)$ and $l(u_2) > l(b)$ then ab coincides with one of the words $v_1, \bar{v}_1, v_2, \bar{v}_2$. Suppose for the sake of definiteness that $ab \equiv v_1$. In this case $a, b \in X_1$ and, because of $l(ab) = l(ba)$, we have $ba \equiv v_1$ or $ba \equiv \bar{v}_1$. The equality $ba \equiv \bar{v}_1$ is impossible since ba is a cyclic permutation of v_1. If $ba \equiv v_1 \equiv ab$, then one of the words a, b must be empty, because v_1 is a simple word. Then $a \in V$ and $b \in V$. Other cases are similar. The lemma is proved.

LEMMA 7. *If $w \equiv \bar{c}\bar{d}cd \in V \setminus (X_1 \cup X_2)$, where $c \not\equiv 1$, $d \not\equiv 1$, then $c, d \in V$.*

PROOF. We have $w \equiv u_1 a_1 \equiv a_2 u_2$, where each of the words u_1, u_2 coincides with one of the words $v_1, \bar{v}_1, v_2, \bar{v}_2$.

If $l(u_1) \leqq l(c)$, then $\bar{c} \equiv u_1 p$, and then $w \equiv u_1 p \bar{d} \bar{p} \bar{u}_1 d$. By Lemma 5, $d \in V$ and, further, $c \in V$. Similarly, if $l(u_2) \leqq l(d)$, then $c \in V$ and $d \in V$. Now we shall show that $l(u_1) > l(c)$ and $l(u_2) > l(d)$ leads to a contradiction. We may assume $u_1 \equiv v_1$, as other cases are similar. Then $\bar{d} \equiv q\bar{d}_1$, where $q \in X_1$, $q \not\equiv 1$, and $w \equiv \bar{c} q \bar{d}_1 c d_1 \bar{q}$, whence $u_2 \equiv v_1$, or $u_2 \equiv \bar{v}_1$. We obtain $c \in X_1$ and $d \in X_1$, and therefore $w \in X_1$. This contradicts the conditions of the lemma. The lemma is proved.

LEMMA 8. *If $w \equiv \bar{b}\bar{c}\bar{d}bcd \in V \setminus (X_1 \cup X_2)$, where the words b, c, d are non-empty, then $b, c, d \in V$.*

PROOF. As $cd \in X$ and $\bar{b}\bar{c} \in X$, then the words $\bar{c}\bar{d}$ and bc are cyclically irreducible. It follows from $cd \in V$ that $\bar{b}\bar{c}\bar{d}b \in V$, whence by Lemma 3, $b \in V$ and $\bar{c}\bar{d} \in V$. Application of Lemma 6 gives $c, d \in V$. Similarly, if $\bar{b}\bar{c} \in V$ then $b, c, d \in V$. We have to prove that $cd \in V$, or $\bar{b}\bar{c} \in V$.

We have $w \equiv u_1 a_1 \equiv a_2 u_2$, where each of the words u_1, u_2 coincides with one of the words $v_1, \bar{v}_1, v_2, \bar{v}_2$. If $l(u_1) \leqq l(b)$, then $\bar{b} \equiv u_1 p$ and $w \equiv u_1 p \bar{c} \bar{d} \bar{p} \bar{u}_1 cd$, and by Lemma 5, $cd \in V$. Similarly, it follows from $l(u_2) \leqq l(d)$ that $\bar{b}\bar{c} \in V$.

Now we shall show that $l(u_1) > l(b)$ and $l(u_2) > l(d)$ leads to a contradiction. We may assume $u_1 \equiv v_1$. Then $\bar{c} \equiv q\bar{c}_1$, where $q \in X_1$, $q \not\equiv 1$. Since $w \equiv \bar{b}\bar{c}\bar{d}bc_1\bar{q}d$ and $l(u_2) > l(d)$, we have $u_2 \equiv v_1$ or $u_2 \equiv \bar{v}_1$. Since $w \notin X_1$, $b \in X_1$, and $d \in X_1$, so $c \equiv c_2 u c_3$, where $u \equiv v_2$, or $u \equiv \bar{v}_2$. We have $w \equiv \bar{b}\bar{c}_3 \bar{u} \bar{c}_2 \bar{d} b c_2 u c_3 d$, whence by Lemma 5, $\bar{b}\bar{c}_3 \in V$ and $c_3 d \in V$. As $\bar{d}b \in X$, by Lemma 4, $b \in V$ in contradiction to $l(u_1) > l(b)$. The lemma is proved.

LEMMA 9. *Let $w \equiv \bar{b}\bar{c}\bar{d}b\bar{e}cde \in V \setminus (X_1 \cup X_2)$, where the words b, c, e are non-empty and, if $d \equiv 1$, then c is cyclically irreducible. Then $b, c, d, e \in V$.*

PROOF. Since $cd \in X$ and $\bar{c}\bar{d} \in X$, the words cd and $\bar{c}\bar{d}$ are cyclically irreducible. It follows from $\bar{b}\bar{c}\bar{d}b \in V$ that $b \in V$, $\bar{c}\bar{d} \in V$ and $\bar{e}cde \in V$. Then $e \in V$ and $cd \in V$ which gives $c \in V$ and $d \in V$. We have to prove that $\bar{b}\bar{c}\bar{d}b \in V$.

We have $w \equiv u_1 a_1 \equiv a_2 u_2$, where each of the words u_1, u_2 coincides with one of the words $v_1, \bar{v}_1, v_2, \bar{v}_2$. We may assume that $u_1 \equiv v_1$. If $u_2 \equiv v_2$ or $u_2 \equiv \bar{v}_2$ then some non-empty word from X_1 is an initial segment of the word $\bar{b}\bar{c}\bar{d}b$, and some non-empty word from X_2 is a final segment of the word $\bar{e}cde$. Then some final segment of $\bar{b}\bar{c}\bar{d}b$ is a word from X_1 and some initial segment of $\bar{e}cde$ is a word from X_2. This implies $\bar{b}\bar{c}\bar{d}b \in V$.

Since $b\bar{e} \in X$ the word w is cyclically irreducible and therefore $u_2 \not\equiv \bar{v}_1$. We have to consider only the case $u_2 \equiv v_1$.

If $l(v_1) \leqq l(b)$, then $\bar{b} \equiv v_1 p$ and $w \equiv v_1 p\bar{c}\bar{d}\bar{p}\bar{v}_1 ecde$, so that by Lemma 5, $\bar{e}cde \in V$ and, further, $\bar{b}\bar{c}\bar{d}b \in V$. Similarly, if $l(v_1) \leqq l(e)$, then $\bar{b}\bar{c}\bar{d}b \in V$.

Now we show that $l(v_1) > l(b)$ and $l(v_1) > l(e)$ leads to a contradiction. It follows from $l(v_1) > l(b)$ and $l(v_1) > l(e)$ that $b \in X_1$, $e \in X_1$ and $\bar{c} \equiv qc_1$, where $q \in X_1$, $q \not\equiv 1$. As $w \not\in X_1$, then $\bar{e}cde \not\in X_1$. This means that $c \equiv c_2uc_3\bar{q}$, or $d \equiv d_1ud_2$, where $u \equiv v_2$, or $u \equiv \bar{v}_2$.

If $c \equiv c_2uc_3\bar{q}$, then $w \equiv \bar{b}q\bar{c}_3\bar{u}\bar{c}_2\bar{d}b\bar{e}c_2uc_3\bar{q}de$. Hence by Lemma 5, $\bar{b}q\bar{c}_3 \in V$. The word w is cyclically irreducible, therefore

$$w^2 \equiv \bar{b}\bar{c}\bar{d}b\bar{e}c_2uc_3\bar{q}de\bar{b}q\bar{c}_3\bar{u}\bar{c}_2\bar{d}b\bar{e}cde \in V,$$

whence $c_3\bar{q}de\bar{b}q\bar{c}_3 \in V$. It follows from $\bar{d}b \in X$ that the word $de\bar{b}$ is cyclically irreducible. By Lemma 3, $c_3\bar{q} \in V$. Then $b \in V$ which contradicts $l(v_1) > l(b)$. Similarly, it follows from $d \equiv d_1ud_2$ that $e \in U$ in contradiction to $l(v_1) > l(e)$. The lemma is proved.

Lemma 10. *If $\bar{b}\bar{c}b\bar{e}ce \in V \setminus (X_1 \cup X_2)$ and $\bar{b}e \in X$, where the words b, c, e are non-empty, then $b, c, e \in V$.*

Proof. Let $c \equiv \bar{c}_1c_2c_1$, where c_2 is cyclically irreducible. By Lemma 9 $c_1b, c_2, c_1e \in V$. Since $\bar{b}\bar{c}_1 \in V$ and $c_1e \in V$, so from $\bar{b}e \in X$ according to Lemma 4 follows $b \in V$, $c_1 \in V$ and $e \in V$. Therefore also $c \in V$. The lemma is proved.

§4. The proof of the Main Lemma

Let the conditions of the Main Lemma be satisfied. Let f_α, $\alpha \in M_1$, be a system of free generators of H_1 and let f_α, $\alpha \in M_2$, be a system of free generators of H_2. We may suppose that elements f_α are chosen in such a way that $h_1\Theta$ and $h_2\Theta$ are cyclically irreducible. Then the condition that $\langle h_1 \rangle$ and $\langle h_2 \rangle$ are isolated means that the words $h_1\Theta$ and $h_2\Theta$ are simple.

Let $V = \langle h_1, h_2 \rangle\Theta$. We have $[g_1, g_2]\Theta \equiv \bar{w}_1w_2w_1 \in V$, where the word w_2 is cyclically irreducible. Then according to Lemma 3, $w_1 \in V$ and $w_2 \in V$. Let $r = \bar{w}_1\xi$, $r_1 = r^{-1}g_1r$ and $r_2 = r^{-1}g_2r$. We have $[r_1, r_2]\Theta \equiv w_2$.

Let the pair of elements $s_1, s_2 \in F$ have the properties

(a) $\langle s_1, s_2 \rangle = \langle r_1, r_2 \rangle$,

(b) $[r_1, r_2] = [s_1, s_2]$,

and the sum of lengths $l(s_1) + l(s_2)$ be the minimal possible for pairs of elements of F satisfying (a) and (b).

The arguments from the proof of theorem 1 of [2] show that the elements s_1, s_2 have also the following properties:

(c) $[s_1, s_2]\Theta \equiv u_1u_2u_3u_4$, where $u_i \not\equiv 1$, $i = 1,2,3,4$, $\overline{s_1\Theta} \equiv u_1q_1$, $\overline{s_2\Theta} \equiv \bar{q}_1u_2q_2$, $s_1\Theta \equiv \bar{q}_2u_3q_3$, $s_2\Theta \equiv \bar{q}_3u_4$,

(d) $2l(q_i) \leqq l(s_j)$, $i = 1,2,3$, $j = 1,2$.

We shall show that $s_1, s_2 \in H_1$, or $s_1, s_2 \in H_2$, or $s_1, s_2 \in \langle h_1, h_2 \rangle$. Then, according to (a), the same will hold for r_1, r_2. Since $r \in \langle h_1, h_2 \rangle$, $\langle h_1, h_2 \rangle \cap H_1 = \langle h_1 \rangle$, $\langle h_1, h_2 \rangle \cap H_2 = \langle h_2 \rangle$ and the subgroup $\langle [r_1, r_2] \rangle$ is isolated, this is enough to complete the proof of the Main Lemma.

If the words q_1, q_2, q_3 are non-empty, then it follows from $s_1\Theta \equiv \bar{q}_1\bar{u}_1 \equiv \bar{q}_2u_3q_3$ and $s_2\Theta \equiv \bar{q}_2\bar{u}_2q_1 \equiv \bar{q}_3u_4$ that for some non-empty q, $q_1 \equiv qp_1$, $q_2 \equiv qp_2$, $q_3 \equiv qp_3$. Then $[s_1, s_2]\Theta \equiv \bar{q}u_1'u_2u_3u_4'q$, where $u_1 \equiv \bar{q}u_1'$, $u_4 \equiv u_4'q$. This contradicts the cyclical irreducibility of $[s_1, s_2]\Theta \equiv [r_1, r_2]\Theta \equiv w_2$. Therefore at least one of the words q_1, q_2, q_3 is empty.

Assume at first that $q_2 \equiv 1$. By the condition (d), for some words u, v we have $s_1\Theta \equiv \bar{q}_1uq_3$ and $s_2\Theta \equiv \bar{q}_3vq_1$. Then $[s_1, s_2]\Theta \equiv \bar{q}_3\bar{u}\bar{v}q_3\bar{q}_1uvq_1$. If $[s_1, s_2]\Theta \in X_1$, then $u, v, q_1, q_3 \in X_1$, whence $s_1\Theta s_2\Theta \in X_1$ and $s_1, s_2 \in H_1$. Similarly, if $[s_1, s_2]\Theta \in X_2$, then $s_1, s_2 \in H_2$.

Let $[s_1, s_2]\Theta \in V \setminus (X_1 \cup X_2)$. If $u \not\equiv 1$ and $v \not\equiv 1$, then according to Lemmas 7, 8, 9, $u, v, q_1, q_3 \in V$, and $s_1, s_2 \in \langle h_1, h_2 \rangle$. If $u \equiv 1$, then because of $[s_1, s_2]\Theta \in X$ and $[s_1, s_2]\Theta \not\equiv 1$ we must have $v \not\equiv 1$. In this case $\overline{s_1\Theta} \equiv \bar{q}_3q_1$. Using Lemmas 7 and 10 we obtain $v, q_1, q_3 \in V$, and again $s_1, s_2 \in \langle h_1, h_2 \rangle$. Similarly, if $v \equiv 1$ then $u \not\equiv 1$ and $s_1, s_2 \in \langle h_1, h_2 \rangle$. It remains to consider the case $q_2 \not\equiv 1$.

TABLE 1

q_1, q_2, q_3	$s_1\Theta$	$s_2\Theta$	$[s_1, s_2]\Theta$
$q_1 \equiv 1$, $q_3 \equiv pq_2$	$\bar{q}_2u_3pq_2$	$\bar{q}_2\bar{p}u_4$	$\bar{q}_2\bar{p}\bar{u}_3q_2\bar{u}_4pu_3u_4$
$q_1 \equiv 1$, $q_3 \equiv q_2$	$\bar{q}_2u_3q_2$	$\bar{q}_2u_4$	$\bar{q}_2\bar{u}_3q_2\bar{u}_4u_3u_4$
$q_1 \equiv 1$, $q_2 \equiv pq_3$, $q_3 \not\equiv 1$	$\bar{q}_3\bar{p}u_3q_3$	$\bar{q}_3\bar{p}\bar{u}_2$	$\bar{q}_3\bar{u}_3pq_3u_2u_3\bar{p}u_2$
$q_1 \equiv q_3 \equiv 1$	$\bar{q}_2u_3$	$\bar{q}_2\bar{u}_2$	$\bar{u}_3q_2u_2u_3\bar{q}_2\bar{u}_2$
$q_1 \equiv pq_2$, $q_3 \equiv 1$	$\bar{q}_2\bar{p}\bar{u}_1$	$\bar{q}_2\bar{u}_2pq_2$	$u_1u_2\bar{p}\bar{u}_1\bar{q}_2\bar{u}_2\bar{u}_2pq_2$
$q_1 \equiv q_2$, $q_3 \equiv 1$	$\bar{q}_2\bar{u}_1$	$\bar{q}_2\bar{u}_2q_2$	$u_1u_2\bar{u}_1\bar{q}_2\bar{u}_2q_2$
$q_2 \equiv pq_1$, $q_1 \not\equiv 1$, $q_3 \equiv 1$	$\bar{q}_1\bar{p}u_3$	$\bar{q}_1\bar{p}\bar{u}_1q_1$	$\bar{u}_3pu_2u_3\bar{q}_1\bar{p}\bar{u}_2q_1$

All the possibilities appearing in the case $q_2 \not\equiv 1$ are collected in Table 1. Here p, whenever it appears, denotes a non-empty word. From the table we see that if $[s_1, s_2]\Theta \in X_1$, then in every case $s_1\Theta \in X_1$ and $s_2\Theta \in X_1$, and therefore $s_1, s_2 \in H_1$. Similarly, if $[s_1, s_2]\Theta \in X_2$, then $s_1, s_2 \in H_2$. If $[s_1, s_2]\Theta \in V \setminus (X_1 \cup X_2)$, then according to Lemmas 8, 9, 10 we obtain $s_1\Theta \in V$, $s_2\Theta \in V$, whence $s_1, s_2 \in \langle h_1, h_2 \rangle$. All the possibilities are considered and therefore the lemma is proved.

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